METRIC SPACES: FINAL EXAM 2018

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Problem 1 (10%). Let $\varepsilon > \varepsilon' > 0$. In a metric space $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$, can an open disk B_ε of larger radius be entirely and strictly contained, $B_\varepsilon \subsetneq B_{\varepsilon'}$, is an open disk $B_{\varepsilon'}$ of smaller radius? (state and prove, e.g., by example)

Problem 2 (20%). Prove that the diameter of every compact metric space $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ is finite.

(By definition, $\mathrm{diam}(\varnothing) = 0$ and $\mathrm{diam}(S) = \sup_{x,y\in S} \mathrm{d}_{\mathcal{X}}(x, y)$ for a non-empty bounded set $S \subseteq \mathcal{X}$.)

Problem 3 (20%). Let $\mathcal{X}$ be a space such that every continuous function $f\colon \mathcal{X} \to \mathbb{R}$ has the following property: if $a < c < b$, $f(x) = a$, and $f(y) = b$, then there exists $z \in \mathcal{X}$ such that $f(z) = c$. Prove $\mathcal{X}$ is connected.

(The set $\mathbb{R}$ is equipped with the standard Euclidean topology.)

Problem 4 (30%=15%+15%). Suppose for every $n \in \mathbb{N}$ that V_n is a non-empty closed subset of a sequentially compact space $\mathcal{X}$ and $V_n \supseteq V_{n+1}$. Prove that

$$\bigcap_{n=1}^{+\infty} V_n \neq \varnothing.$$

- Is this intersection always non-empty if the hypothesis of sequential compactness is discarded? (state and prove, e.g., by counterexample)

Problem 5 (20%). Solve for $x(s)$ the integral equation,

$$x(s) = \tfrac{1}{2} \int_0^1 s \cdot t\, x(t)\, \mathrm{d}t + \tfrac{5}{6} s,$$

by consecutive approximations starting from $x_0(s) = 0$. (In the end, verify by a direct substitution that the solution which you have found satisfies the equation.)

Date: April 4, 2018. Do not postpone your success until June 27. GOOD LUCK!